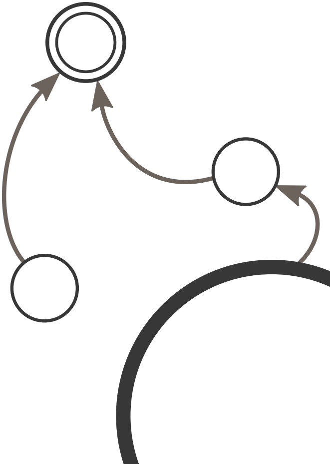
ON THE COMPUTATIONAL POWER OF AFFINE AUTOMATA

LATA2017
Umeå, March 9, 2017



Mika Hirvensalo, **Etienne Moutot**
and Abuzer Yakaryılmaz

DEFINITIONS AND MOTIVATION

DETERMINISTIC FINITE AUTOMATON

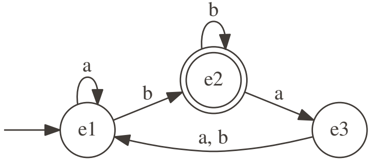


Figure: Example of Deterministic Finite Automaton

$$(\mathbf{M}_x)_{i,j} = 1 \text{ iff } j \xrightarrow{x} i$$

$$\mathbf{M}_a = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

PROBABILISTIC FINITE AUTOMATON

$$(\mathbf{M}_x)_{i,j} = P(j \xrightarrow{x} i)$$

$\mathbf{M}_x$ are stochastic matrices

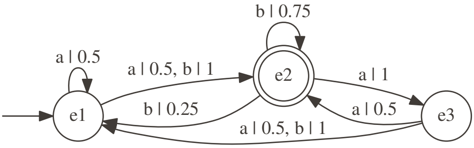


Figure: Example of Probabilistic Finite Automaton

$$\mathbf{M}_a = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_b = \begin{pmatrix} 0 & 0.25 & 1 \\ 1 & 0.75 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

PROBABILISTIC FINITE AUTOMATON

$w = x_1 \ldots x_n$
$\mathbf{M}_w = \mathbf{M}_{x_n} \ldots \mathbf{M}_{x_1}$
P projection associated to accepting states
$\mathbf{v}_0$ initial vector

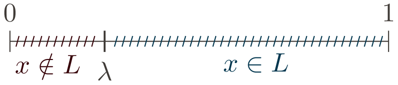
$$f_P(w) = \mathbf{P}\mathbf{M}_w\mathbf{v}_0$$

STOCHASTIC LANGUAGES

Definition (Stochastic language - SL)

$$L = \{w \in \Sigma^* \mid f_P(w) > \lambda\}$$

$\lambda \in (0,1)$ is called the **cutpoint**

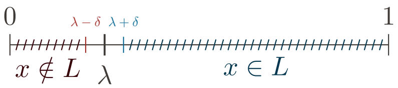


STOCHASTIC LANGUAGES

Definition (Isolated cutpoint - ISL)

$\lambda \in (0,1)$ is an **isolated cutpoint** iff there exist $\delta > 0$ such that

- → $\forall w \in L, f_P(w) \geq \lambda + \delta$
- → $\forall w \notin L, f_P(w) \leq \lambda - \delta$



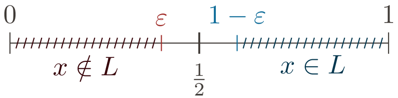
STOCHASTIC LANGUAGES

Definition (Bounded error - BSL)

L is recognized with **bounded error** iff there exists $\varepsilon \in (0, \frac{1}{2})$ such that

→ $\forall w \in L, f_P(w) \geq 1 - \varepsilon$

→ $\forall w \notin L, f_P(w) \leq \varepsilon$



STOCHASTIC LANGUAGES

Theorem

ISL = BSL = REG

GENERAL AUTOMATA ?

Theorem (Jeandel, 2007)

The bounded-error language of **any topological automata** with **compact** set of states and continuous evaluation function is REG.

AFFINE FINITE AUTOMATON

$$A = (E, \Sigma, \{\mathbf{M}_x \mid x \in \Sigma\}, \mathbf{v}_0, E_a)$$

- E finite set of states of P
- Σ finite alphabet
- $\{\mathbf{M}_x\}$ set of transition matrices, all columns sums up to 1
- $\mathbf{v}_0$ initial vector, coordinates sums up to 1
- E_a set of accepting states

AFFINE FINITE AUTOMATON

$w = x_1 \dots x_n$
$\mathbf{M}_w = \mathbf{M}_{x_n} \dots \mathbf{M}_{x_1}$
$\mathbf{P}$ projection associated to accepting states
$\mathbf{v}_0$ initial vector

$$f_A(w) = \frac{|\mathbf{P}\mathbf{M}_w\mathbf{v}_0|}{|\mathbf{M}_w\mathbf{v}_0|} = \frac{\sum_{e_i \in E_a} |(\mathbf{M}_w\mathbf{v}_0)_i|}{\sum_{e_i \in E} |(\mathbf{M}_w\mathbf{v}_0)_i|}$$

AFFINE LANGUAGE

Definition (Affine language - AfL)

$$L = \{w \in \Sigma^* \mid f_P(w) > \lambda\}$$

$\lambda \in (0,1)$ is called the **cutpoint**

BOUNDED ERROR

Definition (Isolated cutpoint - IAfL)

$\lambda \in (0,1)$ is an **isolated cutpoint** iff there exist $\delta > 0$ such that

- $\forall w \in L, f_P(w) \geq \lambda + \delta$
- $\forall w \notin L, f_P(w) \leq \lambda - \delta$

Definition (Bounded error - BAfL)

L is recognized with **bounded error** iff there exists $\varepsilon \in (0, \frac{1}{2})$ such that

- $\forall w \in L, f_P(w) \geq 1 - \varepsilon$
- $\forall w \notin L, f_P(w) \leq \varepsilon$

AFFINE LANGUAGES

Theorem (Díaz-Caron, Yakaryilmaz, 2016)

$\mathrm{IAfL} = \mathrm{BAfL}$

AFFINE LANGUAGES

Theorem (Díaz-Caron, Yakaryilmaz, 2016)

$\mathsf{IAfL} = \mathsf{BAfL} \neq \mathsf{REG}$

STABILITY RESULTS

PRODUCT AND SUM

Theorem

$f, g : \Sigma^* \to [0,1]$ functions of affine automata. fg is affine too.

Theorem

$f, g : \Sigma^* \to [0,1]$ functions of affine automata. $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

$\alpha f + \beta g$ is affine too.

OPERATIONS ON LANGUAGES ?

$L_A, L_B \in$ BAfL, with error bound ε

$L_+ = \{w \mid \frac{1}{2}(f_A(w) + f_B(w)) > \lambda\}$
$L_\times = \{w \mid f_A(w) \times f_B(w) > \lambda\}$

$$\varepsilon < \frac{1}{3} \Rightarrow L_+ \in \text{BAfL}$$

$$L_+ = L_A \cup L_B$$

$$\varepsilon < \frac{1}{3} \Rightarrow L_\times \in \text{BAfL}$$

$$L_\times = L_A \cap L_B$$

ERROR REDUCTION

Theorem (Error Reduction)

Let $L \in$ BAfL. There exists $\mathcal{A}$ such that:

- $\forall w \in L, f_{\mathcal{A}}(w) \geq \frac{3}{4}$
- $\forall w \notin L, f_{\mathcal{A}}(w) \leq \frac{1}{4}$

i.e. $\mathcal{A}$ recognizes L with error bound $\varepsilon = \frac{1}{4}$.

ERROR REDUCTION

Lemma

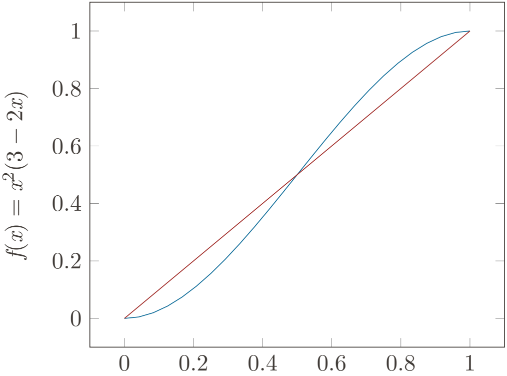
Let f be computed by an affine automaton $\mathcal{A}$. There exists an affine automaton $\mathcal{B}$ such that $f_{\mathcal{B}} = f^2(3-2f)$.

$$\mathbf{B}_x = \mathbf{A}_x \otimes \mathbf{A}_x \otimes \mathbf{A}_x$$

$$\mathbf{v}_0' = \mathbf{v}_0 \otimes \mathbf{v}_0 \otimes \mathbf{v}_0$$

$$E_a' = (E_a \times E_a \times E_a) \cup \left(\overline{E_a} \times E_a \times E_a\right) \cup \left(E_a \times \overline{E_a} \times E_a\right) \cup \left(E_a \times E_a \times \overline{E_a}\right)$$

ERROR REDUCTION



$$\varepsilon \longrightarrow \varepsilon^2(3-2\varepsilon) < \varepsilon$$

OPERATION ON LANGUAGES

Theorem

BAfL is stable under **union**, **intersection** and **complement**.

SOURCE AND LIMITS OF POWER

STATE VECTOR

Theorem

$L \in$ AfL can be recognized by an affine automaton with a **bounded state vector**, at any step of the computation.

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STATE VECTOR

Theorem

$L \in$ AfL can be recognized by an affine automaton with a **bounded state vector**, at any step of the computation.

- ☺ We do not need unbounded space to recognize AfL
- ☹ The proof does not preserve the error gap of there is one

STATE VECTOR (PROOF)

$\mathcal{A}$ initial (unbounded) automaton.
$C = \max_{x,i,j} |(A_x)_{i,j}|$

$$B_x = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \frac{1}{kC}A_x & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline \frac{1}{2} - \frac{1}{2kC} & \cdots & \frac{1}{2} - \frac{1}{2kC} & 1 & 0 \\ \frac{1}{2} - \frac{1}{2kC} & \cdots & \frac{1}{2} - \frac{1}{2kC} & 0 & 1 \end{array}\right)$$

The new function is:

$$f_{\mathcal{B}} = \frac{|PA_w\mathbf{v}_0| + \frac{(kC)^n - 1}{2}}{|A_w\mathbf{v}_0| + (kC)^n - 1}$$

A NON AFFINE LANGUAGE

P a polynomial with nonnegative coefficients and $\deg(P) > 2$,
POLY $= \{a^{P(n)} \mid n \in \mathbb{N}\}$.

PRIME $= \{a^p \mid p \text{ prime}\}$

Adapted from Turakainen 1981:

$$\text{POLY} \notin \text{BAfL}, \text{PRIME} \notin \text{BAfL}$$

LAST SLIDE

Our contributions:

- → Stability results for BAfL
- → No need for non-compact set of states (for AfL)
- → Some languages are not in BAfL

What's next ?

- → BAfL $\neq$ AfL ?

THIS SLIDE IS REALLY THE LAST SLIDE

- Alejandro Díaz-Caro and Abuzer Yakaryilmaz
  »Affine Computation and Affine Automaton«
  Computer Science - Theory and Applications, 2016

- Marcos Villagra and Abuzer Yakaryilmaz
  »Language Recognition Power and Succinctness of Affine Automata«
  Unconventional Computation and Natural Computation, 2016

- Paavo Turakainen
  »On nonstochastic languages and homomorphic images of stochastic languages«
  Information Sciences, 1981

- Emmanuel Jeandel
  »Topological Automata«
  Theory of Computing Systems, 2007

HSRM THEME