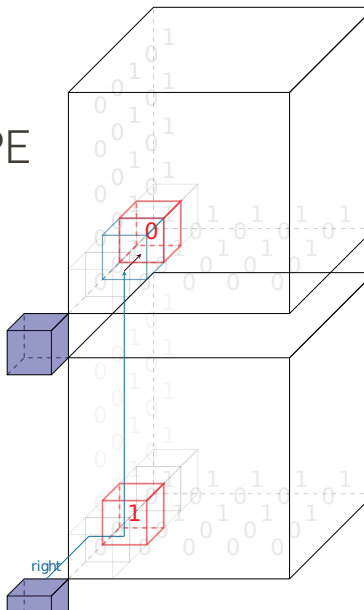


# SLOPES OF 3D SUBSHIFTS OF FINITE TYPE

CSR, June 9, 2018

Etienne Moutot, Pascal Vanier



# UNDECIDABILITY AND ITS FRIENDS

# DECISION PROBLEMS

"Informal" question:

$P$ : "Does  $A(x) = B$ ?"

Set of objects that satisfies a property:

$$P = \{x \mid A(x) = B\}$$

# UNDECIDABLE PROBLEMS

A problem is:

- **Recursively Enumerable** (RE) if a Turing machine accepts its elements
- **Decidable** if it is RE and its complement is RE
- **Undecidable** if not decidable

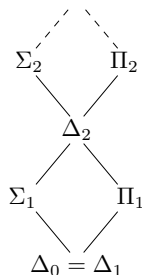
## ARITHMETICAL HIERARCHY

$\Delta_0 = \Sigma_0 = \Pi_0$ : **recursive** problems.

→  $\Sigma_n$ : **recursively enumerable with oracle**  $\Pi_{n-1}$

→  $\Pi_n$ : **complement** of  $\Sigma_n$  or recursively enumerable with oracle  $\Sigma_{n-1}$

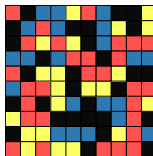
→  $\Delta_n = \Sigma_n \cap \Pi_n$



# SUBSHIFTS AND PERIODICITY

# SUBSHIFTS OF FINITE TYPE (SFT)

→  $\mathcal{A} = \{\blacksquare \blacksquare \blacksquare \blacksquare\}$  a **finite alphabet**

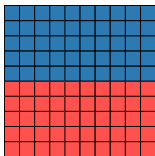


Configuration  $c \in \mathcal{A}^{\mathbb{Z}^2}$

# SUBSHIFTS OF FINITE TYPE (SFT)

→  $\mathcal{A} = \{\blacksquare \square \square \square\}$  a **finite alphabet**

→  $F = \{\square \square \square \square \square \square \square\} \subset \mathcal{A}^{\mathbb{Z}^d}$  a **finite set of finite patterns**



Valid configuration

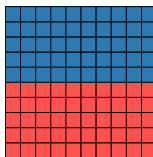


## SUBSHIFTS OF FINITE TYPE (SFT)

→  $\mathcal{A} = \{\blacksquare \blacksquare \blacksquare \blacksquare\}$  a **finite alphabet**

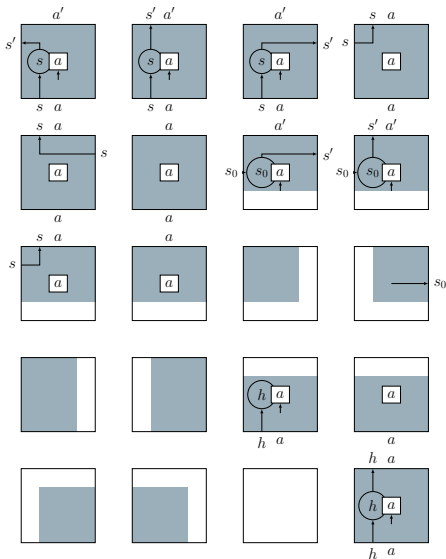
→  $F = \{\blacksquare \blacksquare \blacksquare \blacksquare \blacksquare, \blacksquare \blacksquare \blacksquare \blacksquare \blacksquare, \blacksquare \blacksquare \blacksquare \blacksquare \blacksquare\} \subset \mathcal{A}^{\mathbb{Z}^d}$  a **finite set of finite patterns**

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall p \in F, c \text{ does not contain } p\}$$

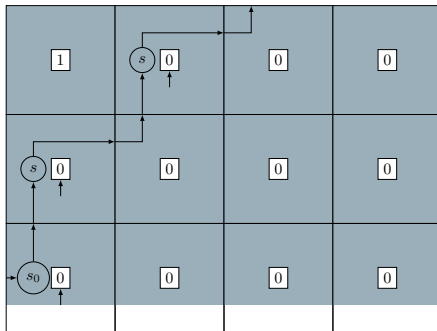


$$c \in X_F$$

# SUBSHIFTS AND COMPUTABILITY



## SUBSHIFTS AND COMPUTABILITY



# UNDECIDABLE PROBLEMS

Emptiness Problem :

"Is  $X_F$  empty ?"

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→ **Undecidable** ( $\in \Sigma_1$ ) for  $d \geq 2$  [Berger, 1964]

# UNDECIDABLE PROBLEMS

Emptiness Problem :

"Is  $X_F$  empty ?"

- **Undecidable** ( $\in \Sigma_1$ ) for  $d \geq 2$  [Berger, 1964]
- **Decidable** for  $d = 1$

## (A) PERIODICITY

A configuration  $c$  is  **$v$ -periodic** if

$$\forall x \in \mathbb{Z}^2, c(x) = c(x + v)$$

A configuration without periodicity vector is **aperiodic**.

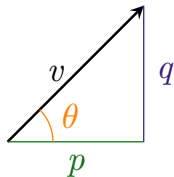
# SLOPES OF PERIODICITY



## 2D CASE: DEFINITIONS

$$v = (p, q).$$

The **slope** of  $v$  is  $\theta = \frac{p}{q}$ .



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$c$  1-periodic with slope  $\theta$ :

$c$  has **slope of periodicity**  $\theta$

$S_X = \{\theta \text{ slope of periodicity of } c \mid c \in X\}$  is the **set of slopes** of the SFT  $X$

# 2D CASE: DEFINITIONS

$$\mathcal{A} = \{\text{red square}, \text{blue square}\} \quad F = \{\text{red blue}, \text{blue red}, \text{red blue red}, \text{blue red blue}\}$$

$$X_F = \left\{ \begin{array}{|c|} \hline \text{blue grid} \\ \hline \text{red grid} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{red grid} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{blue grid} \\ \hline \end{array} \right\}$$

$$S_{X_F} = \{0\}$$

## 2D CASE: DEFINITIONS

$$v = (p, q).$$

The **slope** of  $v$  is  $\theta = \frac{p}{q}$ .

$c$  1-periodic with slope  $\theta$ :

$c$  has **slope of periodicity**  $\theta$

$S_X = \{\theta \text{ slope of periodicity of } c \mid c \in X\}$  is the **set of slopes** of the SFT  $X$

$S = \{S_X \mid X \text{ a 2D SFT}\}$  is the set of all set of slopes

## 2D CASE: MAIN THEOREM

### Theorem [Jeandel, Vanier 2010]

In dimension 2, The problem

"Does SFT  $X$  have slope  $\theta$  ?"

is  $\Sigma_1$ -complete.

## 2D CASE: MAIN THEOREM

Theorem [Jeandel, Vanier 2010]

In dimension 2,  $S = \Sigma_1 \cap P(\mathbb{Q} \cup \infty)$ .

## 3D CASE: CONJECTURE

### Conjecture [Jeandel, Vanier 2010]

In dimension **3**, The problem

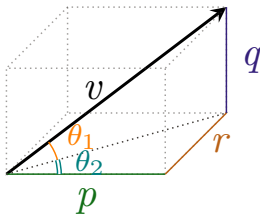
"Does SFT  $X$  have slope  $\theta$  ?"

is  $\Sigma_2$ -complete.

## 3D CASE: DEFINITIONS

$$v = (p, q, r).$$

The **slope** of  $v$  is  $(\theta_1, \theta_2) = \left(\frac{p}{q}, \frac{p}{r}\right)$ .





## 3D CASE: DEFINITIONS

$$v = (p, q, r).$$

The **slope** of  $v$  is  $(\theta_1, \theta_2) = (\frac{p}{q}, \frac{p}{r})$ .

$c$  1-periodic with slope  $\theta$ :

$c$  has **slope of periodicity**  $\theta$

$S_X = \{\theta \text{ slope of periodicity of } c \mid c \in X\}$  is the **set of slopes** of the SFT  $X$

$S = \{S_X \mid X \text{ a 3D SFT}\}$  is the set of all set of slopes

## 3D CASE

### Theorem

In dimension 3,  $S \supseteq \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$ .

### Theorem [Grandjean, Hellouin, Vanier 2018]

In dimension 3,  $S \subseteq \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$ .

## COMPLEXITY GAP: INTUITION

→  $X$  a 2D SFT

→  $c \in X$

→  $\theta \in \mathbb{Q} \cup \infty$

⇒  $\exists Y_{c,\theta}$  of dimension 1 such that:

"Is  $c$  periodic along  $\theta$ ?"

⇔

"Is  $Y_{c,\theta}$  empty?"

**Decidable**

## COMPLEXITY GAP: INTUITION

→  $X$  a 2D SFT

→  $c \in X$

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⇒  $\exists Y_{c,\theta}$  of dimension 1 such that:

"Is  $\theta$  slope of  $X$ ?"

⇔

" $\exists c \in X$  such that  $Y_{c,\theta}$  empty?"

∈  $\Sigma_1$

## COMPLEXITY GAP: INTUITION

→  $X$  a **3D** SFT

→  $c \in X$

→  $\theta \in (\mathbb{Q} \cup \infty)^2$

⇒  $\exists Y_{c,\theta}$  of dimension **2** such that:

"Is  $c$  periodic along  $\theta$ ?"

⇔

"Is  $Y_{c,\theta}$  empty?"

∈  $\Sigma_1$

## COMPLEXITY GAP: INTUITION

→  $X$  a **3D** SFT

→  $c \in X$

→  $\theta \in (\mathbb{Q} \cup \infty)^2$

⇒  $\exists Y_{c,\theta}$  of dimension **2** such that:

"Is  $\theta$  slope of  $X$ ?"

⇔

" $\exists c \in X$  such that  $Y_{c,\theta}$  empty?"

∈  $\Sigma_2$

## PROOF IDEAS

## Theorem

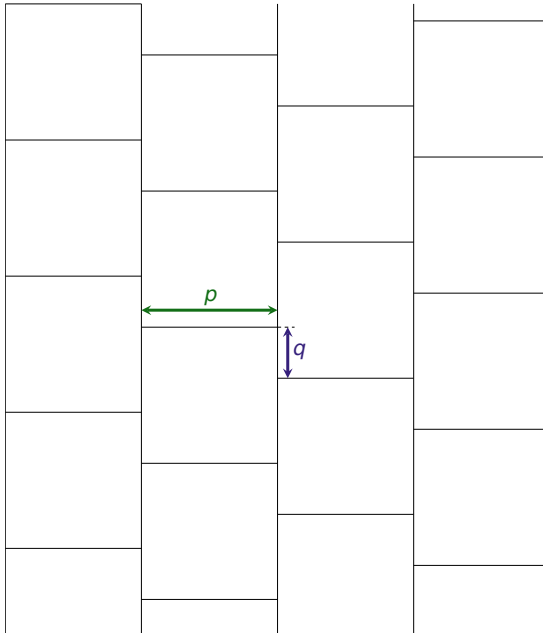
Let  $R \in \Sigma_2 \cap P(\mathbb{Q} \cup \infty)$ . Then there exists an SFT  $X$  such that  $R = S_X$ .

Let  $M$  be a Turing machine  $\Sigma_2$  such that  $R = \{\theta \mid M \text{ accepts } \theta\}$ .

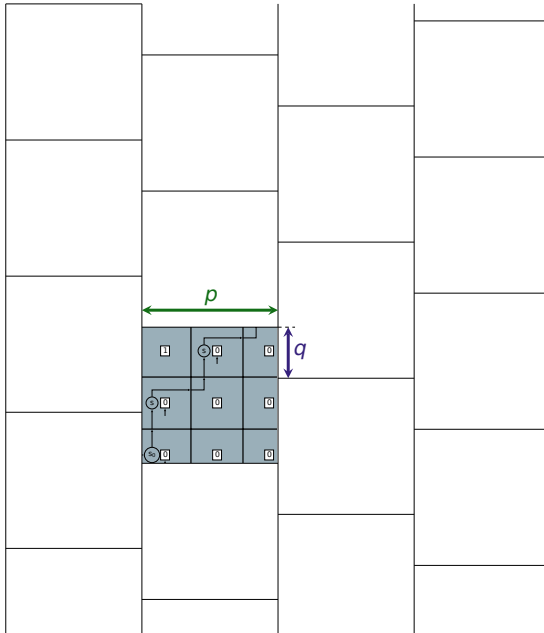
**Goal:** Construct  $X$  such that  $S_X = R$ .

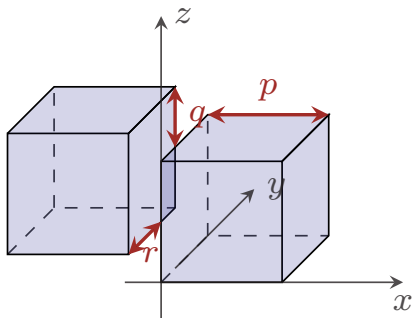
$\Leftrightarrow$

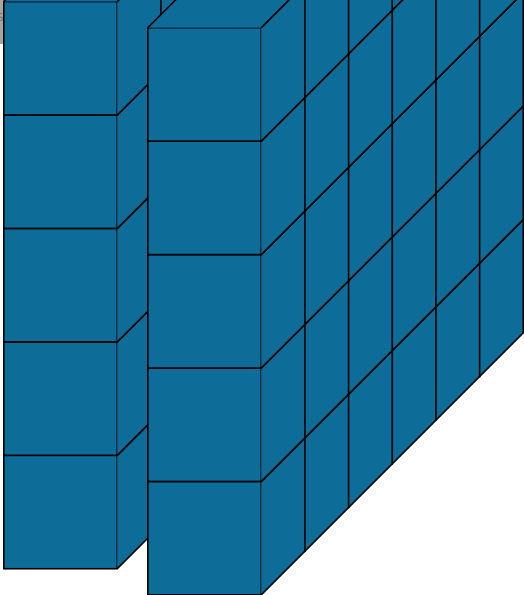
Any 1-periodic configuration of  $X$  has slope  $\theta = (\frac{p}{q}, \frac{p}{r}) \in R$ .

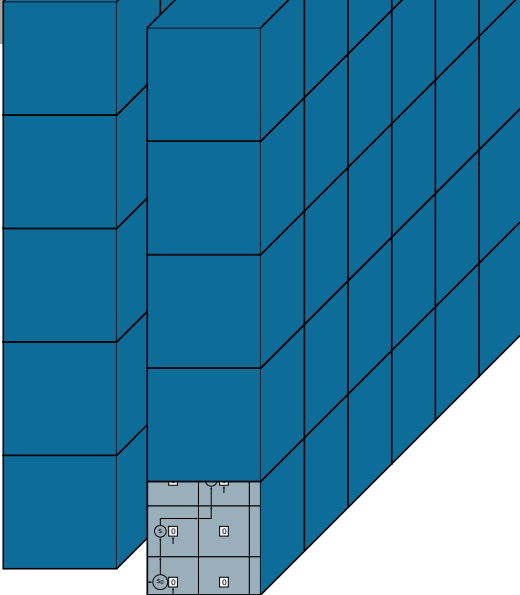


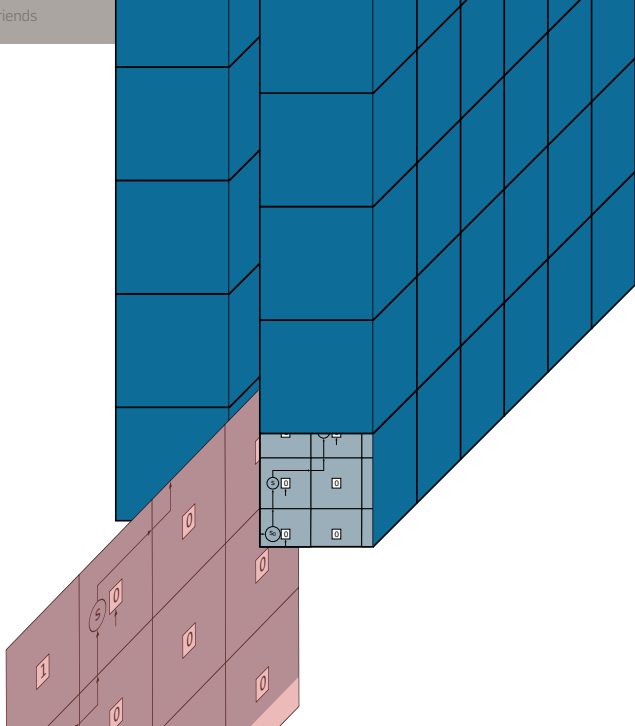












$$X = B \times B' \times B'' \times C \times W \times P \times S \times T_O \times T_M \times A$$

- $B$ ,  $B'$  and  $B''$  create cuboids with pieces of aperiodic background in them
- $C$  forces cubes to appear
- $W$  creates a periodicity vector, and writes the input in the cubes
- $P$  reduces the size of the output
- $S$  synchronizes aperiodic background between cubes
- $T_O$  encodes the oracle  $\Pi_1$
- $T_M$  encodes the Turing machine  $\Sigma_2$
- $A$  ensures the existence of configurations with unique periodicity

## WHAT NEXT ?

- $\Sigma_2$ -hardness seems to work for higher dimensions
- ... But not the proof of  $\in \Sigma_2$ .

Thank you !



# HSRM THEME



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