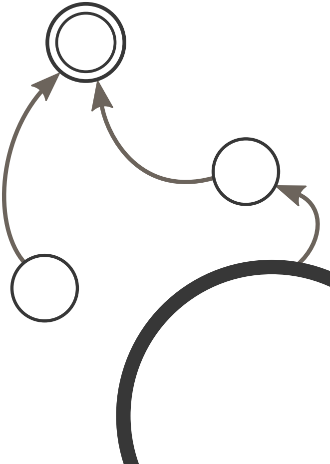
ON THE COMPUTATIONAL POWER OF AFFINE AUTOMATA



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DEFINITIONS AND MOTIVATION

DETERMINISTIC FINITE AUTOMATON

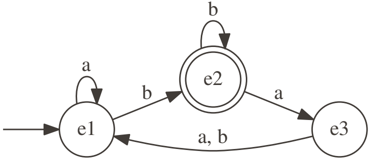


Figure: Example of Deterministic Finite Automaton

$$(\mathbf{M}_x)_{i,j} = 1 \text{ iff } j \xrightarrow{x} i$$

$$\mathbf{M}_a = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

PROBABILISTIC FINITE AUTOMATON

$$(\mathbf{M}_x)_{i,j} = P(j \xrightarrow{x} i)$$

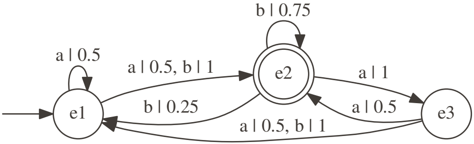


Figure: Example of Probabilistic Finite Automaton

$\mathbf{M}_x$ are stochastic matrices

$$\mathbf{M}_a = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_b = \begin{pmatrix} 0 & 0.25 & 1 \\ 1 & 0.75 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

PROBABILISTIC FINITE AUTOMATON

$w = x_1 \ldots x_n$
$\mathbf{M}_w = \mathbf{M}_{x_n} \ldots \mathbf{M}_{x_1}$
$\mathbf{P}$ projection associated to accepting states
$\mathbf{v}_0$ initial vector

$$f_P(w) = \mathbf{P}\mathbf{M}_w\mathbf{v}_0$$

STOCHASTIC LANGUAGES

Definition (Stochastic language - SL)

$$L = \{w \in \Sigma^* \mid f_P(w) > \lambda\}$$

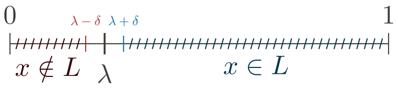
$\lambda \in (0,1)$ is called the **cutpoint**

STOCHASTIC LANGUAGES

Definition (Isolated cutpoint - ISL)

$\lambda \in (0,1)$ is an **isolated cutpoint** iff there exist $\delta > 0$ such that

- → $\forall w \in L, f_P(w) \geq \lambda + \delta$
- → $\forall w \notin L, f_P(w) \leq \lambda - \delta$

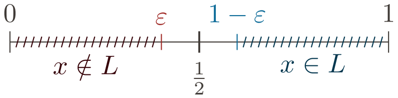


STOCHASTIC LANGUAGES

Definition (Bounded error - BSL)

L is recognized with **bounded error** iff there exists $\varepsilon \in (0, \frac{1}{2})$ such that

- → $\forall w \in L, f_P(w) \geq 1 - \varepsilon$
- → $\forall w \notin L, f_P(w) \leq \varepsilon$



STOCHASTIC LANGUAGES

Theorem

ISL $=$ BSL $=$ REG

GENERAL AUTOMATA ?

Theorem (Jeandel, 2007)

The bounded-error language of any topological automata with compact set of states and continuous evaluation function is REG.

AFFINE FINITE AUTOMATON

$$A = (E, \Sigma, \{\mathbf{M}_x \mid x \in \Sigma\}, \mathbf{v}_0, E_a)$$

E finite set of states of P

Σ finite alphabet

$\{\mathbf{M}_x\}$ set of transition matrices, all columns sums up to 1

$\mathbf{v}_0$ initial vector, coordinates sums up to 1

E_a set of accepting states

AFFINE FINITE AUTOMATON

$w = x_1 \dots x_n$
$\mathbf{M}_w = \mathbf{M}_{x_n} \dots \mathbf{M}_{x_1}$
$\mathbf{P}$ projection associated to accepting states
$\mathbf{v}_0$ initial vector

$$f_A(w) = \frac{|\mathbf{P}\mathbf{M}_w\mathbf{v}_0|}{|\mathbf{M}_w\mathbf{v}_0|} = \frac{\sum_{e_i \in E_a} |(\mathbf{M}_w\mathbf{v}_0)_i|}{\sum_{e_i \in E} |(\mathbf{M}_w\mathbf{v}_0)_i|}$$

AFFINE LANGUAGE

Definition (Affine language - AfL)

$$L = \{w \in \Sigma^* \mid f_P(w) > \lambda\}$$

$\lambda \in (0,1)$ is called the **cutpoint**

BOUNDED ERROR

Definition (Isolated cutpoint - IAfL)

$\lambda \in (0,1)$ is an **isolated cutpoint** iff there exist $\delta > 0$ such that

- $\forall w \in L, f_P(w) \geq \lambda + \delta$
- $\forall w \notin L, f_P(w) \leq \lambda - \delta$

Definition (Bounded error - BAfL)

L is recognized with **bounded error** iff there exists $\varepsilon \in (0, \frac{1}{2})$ such that

- $\forall w \in L, f_P(w) \geq 1 - \varepsilon$
- $\forall w \notin L, f_P(w) \leq \varepsilon$

AFFINE LANGUAGES

Theorem (Díaz-Caron, Yakaryilmaz, 2016)

IAfL = BAfL

AFFINE LANGUAGES

Theorem (Díaz-Caron, Yakaryilmaz, 2016)

$\mathrm{IAfL} = \mathrm{BAfL} \neq \mathrm{REG}$

STABILITY RESULTS

PRODUCT AND SUM

Theorem

$f, g : \Sigma^* \to [0,1]$ functions of affine automata. fg is affine too.

Theorem

$f, g : \Sigma^* \to [0,1]$ functions of affine automata. $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

$\alpha f + \beta g$ is affine too.

OPERATIONS ON LANGUAGES ?

$L_A, L_B \in$ BAfL, with error bound ε

$L_+ = \{w \mid \frac{1}{2}(f_A(w) + f_B(w)) > \lambda\}$
$L_\times = \{w \mid f_A(w) \times f_B(w) > \lambda\}$

$$\varepsilon < \frac{1}{3} \Rightarrow L_+ \in \text{BAfL}$$

$$L_+ = L_A \cup L_B$$

$$\varepsilon < \frac{1}{3} \Rightarrow L_\times \in \text{BAfL}$$

$$L_\times = L_A \cap L_B$$

ERROR REDUCTION

Theorem (Error Reduction)

Let $L \in$ BAfL*. There exists* $\mathcal{A}$ *such that:*

- $\forall w \in L, f_{\mathcal{A}}(w) \geq \frac{3}{4}$
- $\forall w \notin L, f_{\mathcal{A}}(w) \leq \frac{1}{4}$

i.e. $\mathcal{A}$ *recognizes* L *with error bound* $\varepsilon = \frac{1}{4}$.

ERROR REDUCTION

Lemma

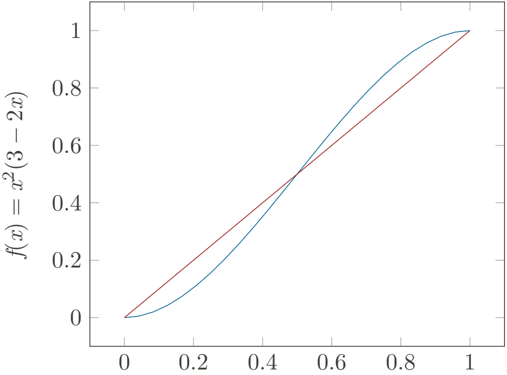
Let f be computed by an affine automaton $\mathcal{A}$. There exists an affine automaton $\mathcal{B}$ such that $f_{\mathcal{B}} = f^2(3-2f)$.

$$\mathbf{B}_x = \mathbf{A}_x \otimes \mathbf{A}_x \otimes \mathbf{A}_x$$

$$\mathbf{v}'_0 = \mathbf{v}_0 \otimes \mathbf{v}_0 \otimes \mathbf{v}_0$$

$$E'_a = (E_a \times E_a \times E_a) \cup \left(\overline{E_a} \times E_a \times E_a\right) \cup \left(E_a \times \overline{E_a} \times E_a\right) \cup \left(E_a \times E_a \times \overline{E_a}\right)$$

ERROR REDUCTION



$$\varepsilon \longrightarrow \varepsilon^2(3-2\varepsilon) < \varepsilon$$

OPERATION ON LANGUAGES

Theorem

BAfL *is stable under union, intersection and complement.*

SOURCE AND LIMITS OF POWER

STATE VECTOR

Theorem

$L \in$ AfL *can be recognized by an affine automaton with a bounded state vector, at any step of the computation.*

STATE VECTOR

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$L \in$ AfL can be recognized by an affine automaton with a bounded state vector, at any step of the computation.

☺ We do not need unbounded space to recognize AfL

STATE VECTOR

Theorem

$L \in$ AfL *can be recognized by an affine automaton with a bounded state vector, at any step of the computation.*

- ☺ We do not need unbounded space to recognize AfL
- ☹ The proof does not preserve the error gap of there is one

STATE VECTOR (PROOF)

$\mathcal{A}$ initial (unbounded) automaton.
$C = \max_{x,i,j} |(A_x)_{i,j}|$

$$B_x = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \frac{1}{kC}A_x & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline \frac{1}{2} - \frac{1}{2kC} & \cdots & \frac{1}{2} - \frac{1}{2kC} & 1 & 0 \\ \frac{1}{2} - \frac{1}{2kC} & \cdots & \frac{1}{2} - \frac{1}{2kC} & 0 & 1 \end{array}\right)$$

The new function is:

$$f_{\mathcal{B}} = \frac{|PA_w\mathbf{v}_0| + \frac{(kC)^n - 1}{2}}{|A_w\mathbf{v}_0| + (kC)^n - 1}$$

A NON AFFINE LANGUAGE

P a polynomial with nonnegative coefficients and $\deg(P) > 2$,
POLY $= \{a^{P(n)} \mid n \in \mathbb{N}\}$.

PRIME $= \{a^p \mid p$ prime$\}$

Adapted from Turakainen 1981:

$$\text{POLY} \notin \text{BAfL}, \text{PRIME} \notin \text{BAfL}$$

LAST SLIDE

Our contributions:

- → Stability results for BAfL
- → No need for non-compact set of states (for AfL)
- → Some languages are not in BAfL

What's next ?

- → BAfL $\neq$ AfL ?

THIS SLIDE IS REALLY THE LAST SLIDE

- Alejandro Díaz-Caro and Abuzer Yakaryilmaz
  »Affine Computation and Affine Automaton«
  Computer Science - Theory and Applications, 2016
- Marcos Villagra and Abuzer Yakaryilmaz
  »Language Recognition Power and Succinctness of Affine Automata«
  Unconventional Computation and Natural Computation, 2016
- Paavo Turakainen
  »On nonstochastic languages and homomorphic images of stochastic languages«
  Information Sciences, 1981
- Emmanuel Jeandel
  »Topological Automata«
  Theory of Computing Systems, 2007

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