DECIDABILITY AND PERIODICITY OF LOW COMPLEXITY TILINGS

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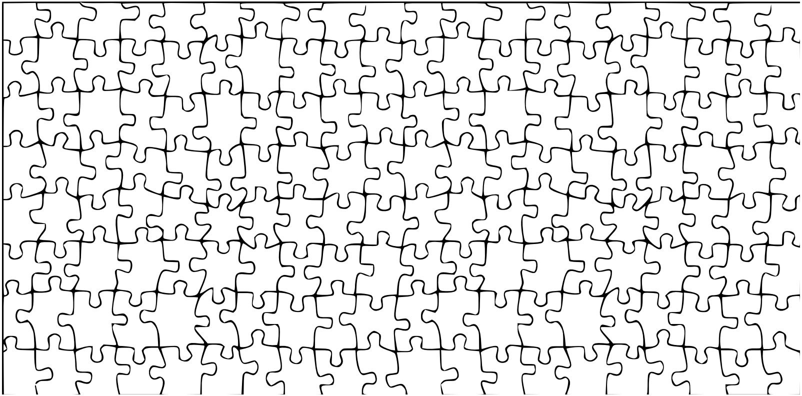
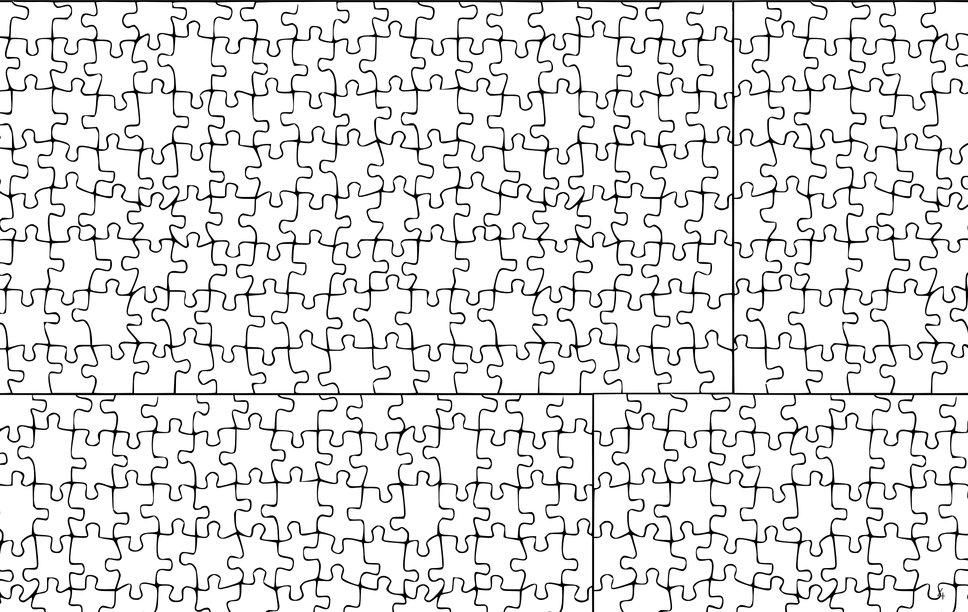
TILINGS

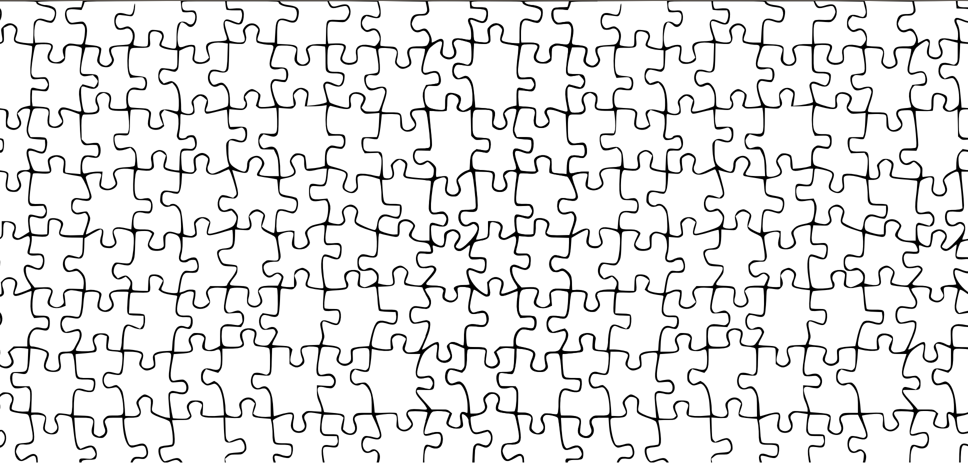
JIGSAW PUZZLES



INFINITE JIGSAW PUZZLES?



MORE INTERESTING INFINITE JIGSAW PUZZLES



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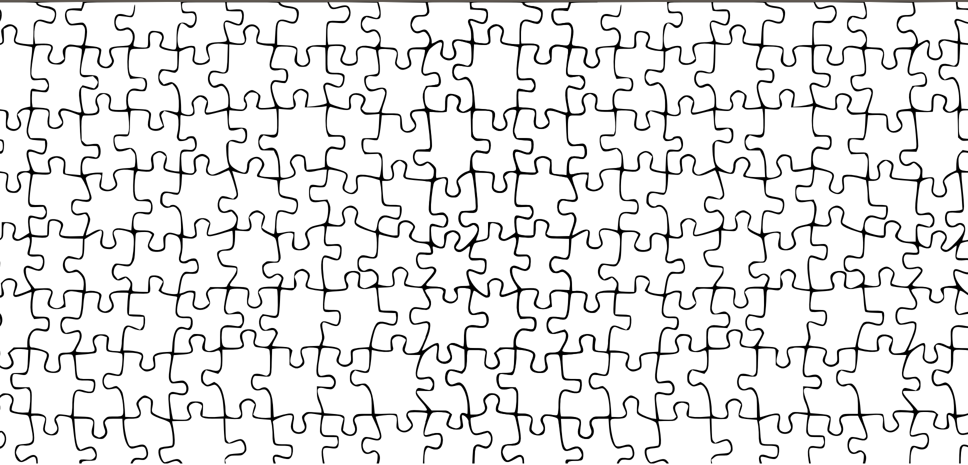


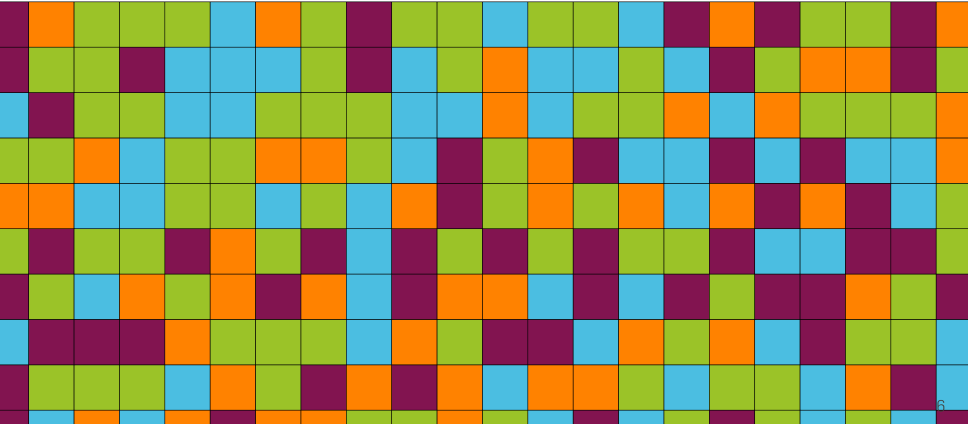
Can we make an infinite puzzle ?

What patterns can appear ?

CONFIGURATION

$\mathcal{A} = \{$  $\}$

$$c \in \mathcal{A}^{\mathbb{Z}^2}$$

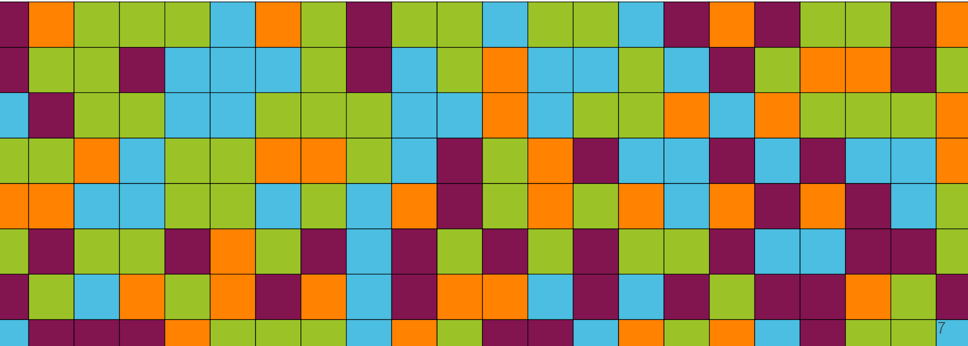


SUBSHIFT

$\mathcal{A} = \{$  $\}$

$F = \{$  $\}$ set of *forbidden patterns*

$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall m \in F, m$ does **not** appear in $c\}$



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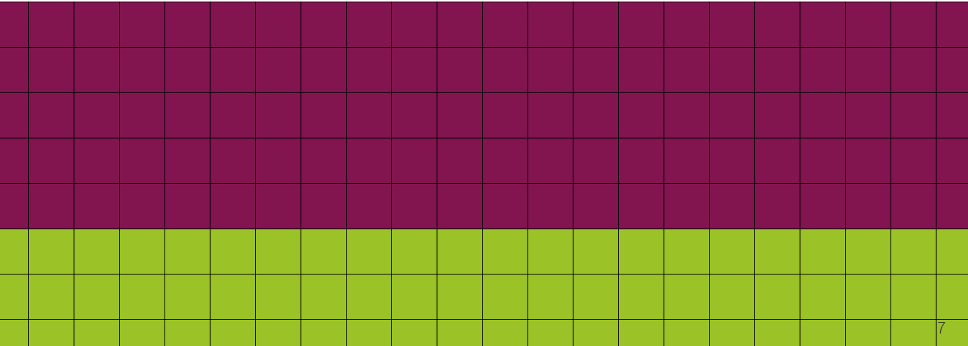

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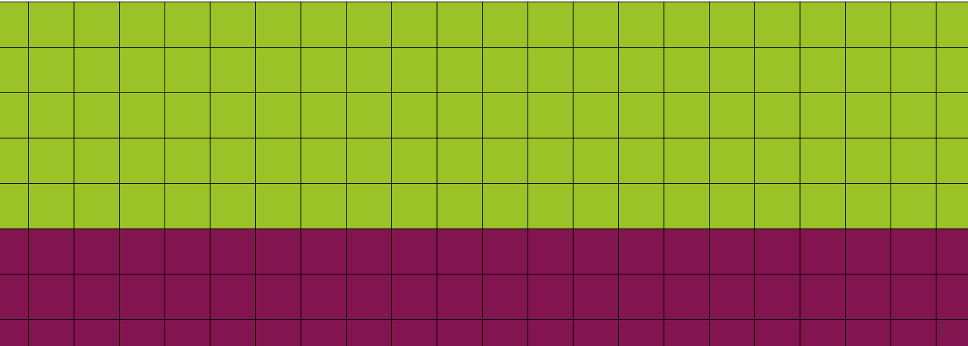

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$F = \{$  $\}$ set of *forbidden patterns*

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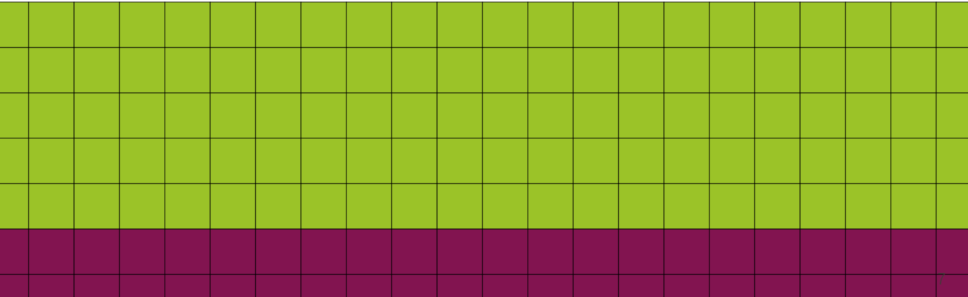
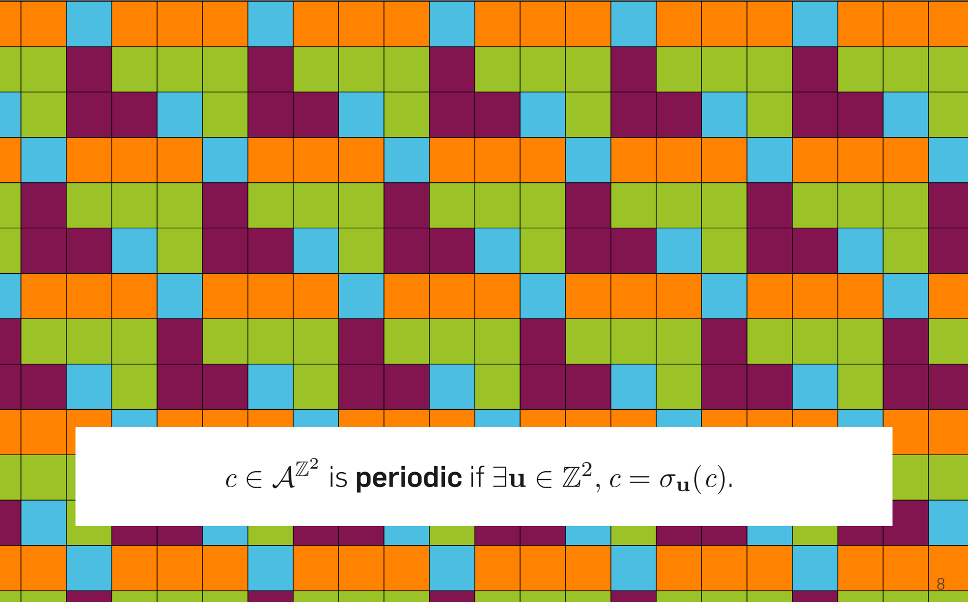
is a **Subshift of Finite Type (SFT)** if F is finite



(A)PERIODICITY



$c \in \mathcal{A}^{\mathbb{Z}^2}$ is **periodic** if $\exists \mathbf{u} \in \mathbb{Z}^2, c = \sigma_{\mathbf{u}}(c)$.

DOMINO PROBLEM (OR WHY WE CARE ABOUT APERIODICITY)

Input:

Subshift of Finite Type (SFTs):

- → Finite Alphabet
- → Finite set of forbidden patterns

Domino Problem:

Is there a coloring of $\mathbb{Z}^2$ with no forbidden patterns ?

Is X_F non-empty ?

DECIDABILITY

Theorem

In dimension 1 ($\mathbb{Z}$), the Domino Problem is **decidable**.

(UN)DECIDABILITY

Theorem (Berger, 1964)

In dimension 2 ($\mathbb{Z}^2$), the Domino Problem is **undecidable**.

DP: DECIDING EMPTINESS

Lemma

There is a semi-algorithm deciding if $X_F = \emptyset$

1. `i=1`
2. `If all colorings of` B_i `contain` $p \in F$:

$$X_F = \emptyset$$

3. `Else i=i+1 and repeat`

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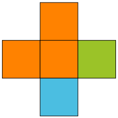

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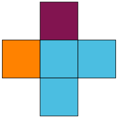

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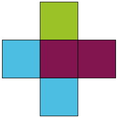

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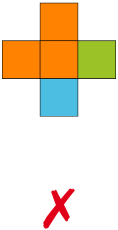

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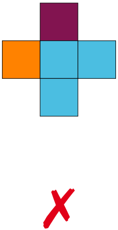

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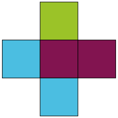

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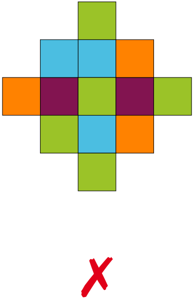

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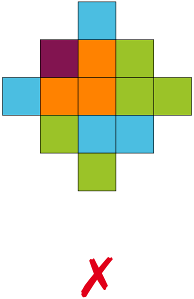

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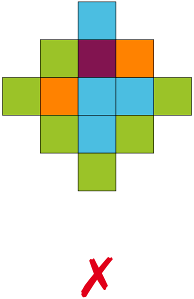

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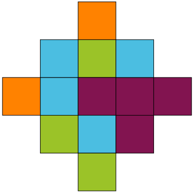

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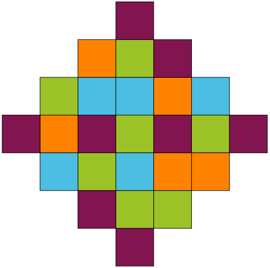

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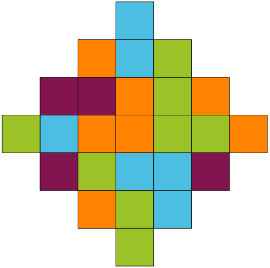

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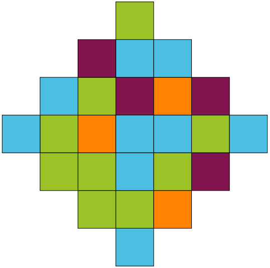

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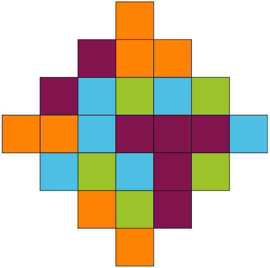

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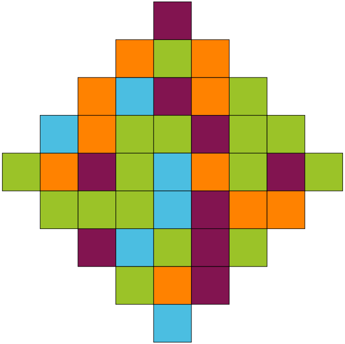

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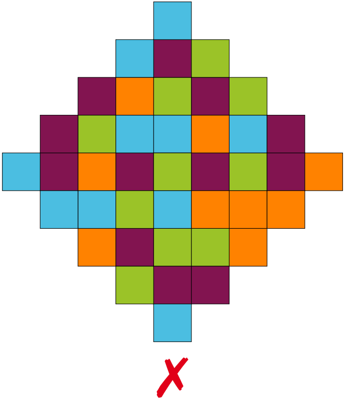

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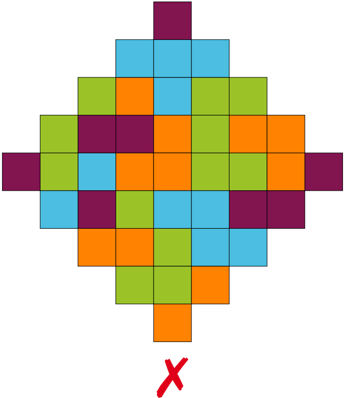

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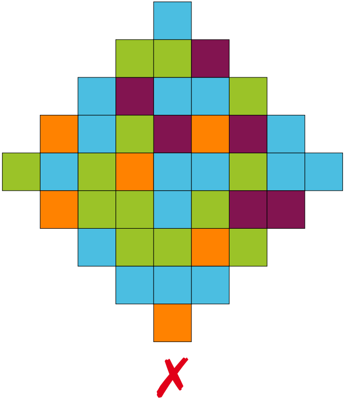

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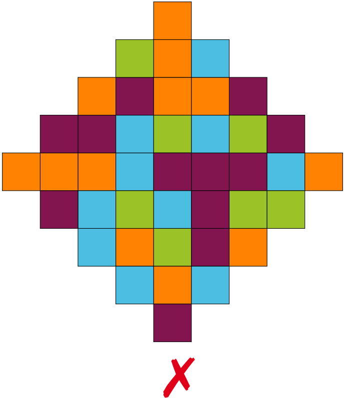

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✗: Impossible to color one ball: $X_F = \emptyset$.

DP: DECIDING NON-EMPTINESS (IMPOSSIBLE)

Building **periodic** configurations

1. `i=1`
2. `If a colorings of` B_i `with no` $p \in F$ `has a period:`

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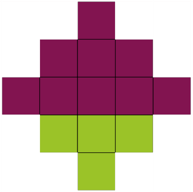

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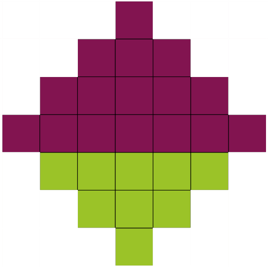

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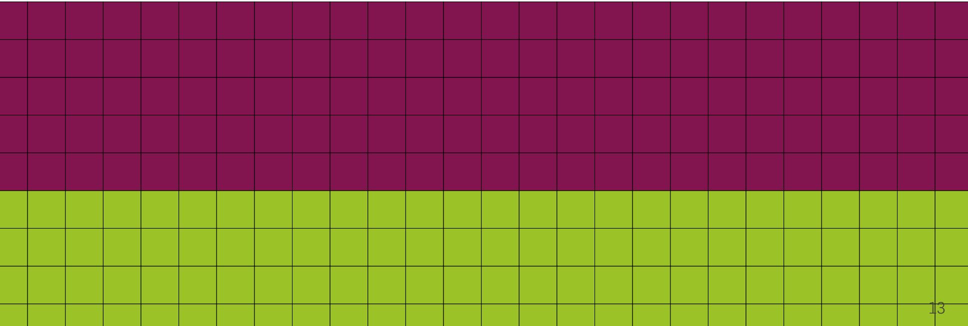

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But what if $X_F \neq \emptyset$ but has no periodic configuration ?

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But what if $X_F \neq \emptyset$ but has no periodic configuration ?

Theorem (Berger, 1964)

There is no semi-algorithm deciding if $X_F \neq \emptyset$.

APERIODIC SFT

Key element of the undecidability: **Aperiodic SFT:**

SFT X such that all $c \in X$ are aperiodic.

NIVAT'S CONJECTURE

NIVAT'S CONJECTURE

Nivat's conjecture: linking **periodicity** and **patterns**

DIMENSION 1

$\mathcal{A} = \{$  $\}$

$w \in \mathcal{A}^{\mathbb{Z}}$

Complexity of w:

$P_w(n) =$ number of patterns of size n

$$P_w(1) = 4$$

$$P_w(2) \geq 11$$

$$P_w(4) \geq 18$$

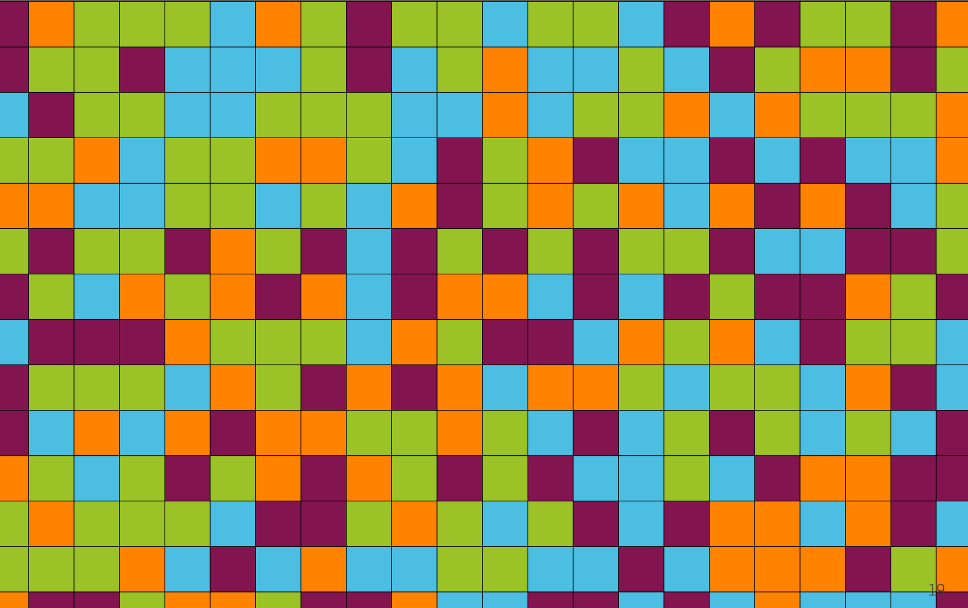
DIMENSION 1

Theorem (Morse, Hedlund, 1938)

$$\forall w \in \mathcal{A}^{\mathbb{Z}},$$

$$\exists n > 0,\ P_w(n) \leq n \quad \Rightarrow \quad w \textit{ periodic}$$

DIMENSION 2



DIMENSION 2

$P_c(m,n) =$ number of rectangular patterns of size $m \times n$

Low complexity:

$$\exists n, m > 0, \quad P_c(m,n) \leq mn$$

2D: NIVAT'S CONJECTURE

Conjecture (Nivat, 1997)

$$\forall c \in \mathcal{A}^{\mathbb{Z}^2},$$

$$\exists m, n > 0, P_c(m, n) \leq mn \quad \Rightarrow \quad c \text{ periodic}$$

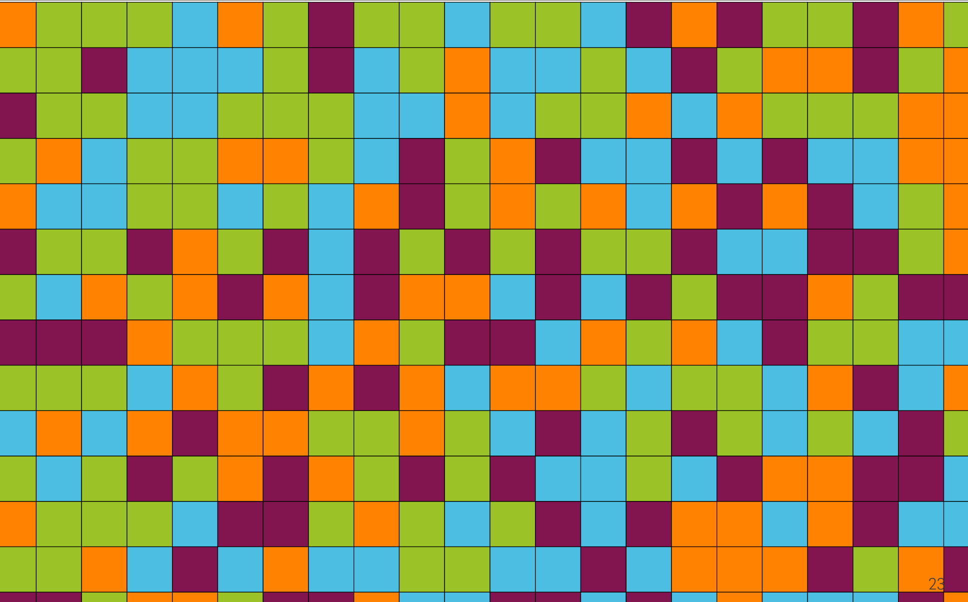
GETTING CLOSE TO NIVAT'S CONJECTURE

- → $P_c(m,n) \leq \frac{mn}{144}$ [Epifanio, Koskas & Mignosi, 2003]
- → $P_c(m,n) \leq \frac{mn}{16}$ [Quas & Zamboni, 2004]

  $\dots$
- → $P_c(m,n) \leq \frac{mn}{2}$ [Cyr & Kra, 2015]

ALGEBRAIC TOOLS

CONFIGURATIONS ARE (LAURENT) SERIES



CONFIGURATIONS ARE (LAURENT) SERIES

3	1	1	1	0	3	1	2	1	1	0	1	1	0	2	3	2	1	1	2	3	1
1	1	2	0	0	0	1	2	0	1	3	0	0	1	0	2	1	3	3	2	1	3
2	1	1	0	0	1	1	1	0	0	3	0	1	1	3	0	3	1	1	1	3	3
1	3	0	1	1	3	3	1	0	2	1	3	2	0	0	2	0	2	0	0	3	3
3	0	0	1	1	0	1	0	3	2	1	3	1	3	0	3	2	3	2	0	1	3
2	1	1	2	3	1	2	0	2	1	2	1	2	1	1	2	0	0	2	2	1	3
1	0	3	1	3	2	3	0	2	3	3	0	2	0	2	1	2	2	3	1	2	2
2	2	2	3	1	1	1	0	3	1	2	2	0	3	1	3	0	2	1	1	0	0
1	1	1	0	3	1	2	3	2	3	0	3	3	1	0	1	1	0	3	2	0	3
0	3	0	3	2	3	3	1	1	3	1	0	2	0	1	2	1	0	1	0	2	1
1	0	1	2	1	3	2	3	1	2	1	2	0	0	1	0	2	3	3	2	2	0
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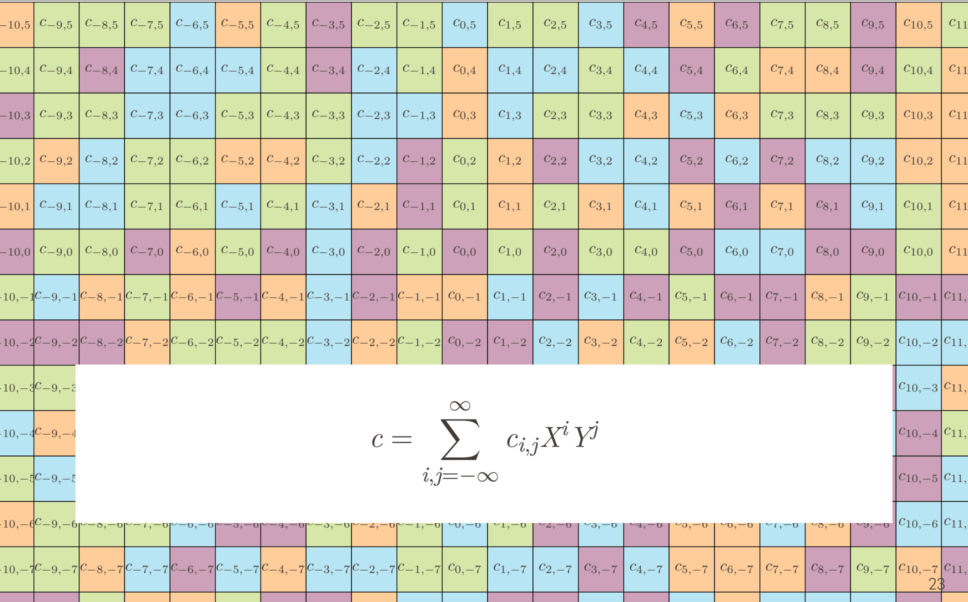

$$c = \sum_{i,j=-\infty}^{\infty} c_{i,j} X^i Y^j$$

OPERATIONS: SUM

$$c + d = \sum_{i,j=-\infty}^{\infty} (c_{i,j} + d_{i,j})\, X^i Y^j$$

Formal sum $\leftrightarrow$ Sum of configurations

OPERATIONS: MULTIPLICATION

$$X^a Y^b c = \sum_{i,j=-\infty}^{\infty} c_{i,j} X^{i+a} Y^{j+b}$$

Multiplication by $X^a Y^b$ $\leftrightarrow$ Translation of vector (a, b)

EXPRESSING PERIODICITY

$$(X^a Y^b - 1)c = 0$$

$$\Leftrightarrow$$

$$X^a Y^b c = c$$

$$\Leftrightarrow$$

c periodic of period (a, b)

ALGEBRA IS COMING

$$\text{Ann}(c) = \{p \mid pc = 0\}$$

$$c \text{ periodic } \Leftrightarrow \exists a, b \in \mathbb{Z}, (X^a Y^b - 1) \in \text{Ann}(c)$$

Ann(c) is a **polynomial ideal**:

- → $0 \in I$
- → $f, g \in I \Rightarrow f + g \in I$
- → $f \in I$ and h any polynomial $\Rightarrow fh \in I$

WARMING UP (MORE)

c of low complexity

Theorem (Kari, Szabados, 2015)

$\exists p \neq 0 \in \mathrm{Ann}(c)$

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Theorem (Kari, Szabados, 2015)

$\exists a_1, b_1, a_2, b_2 \ldots, a_r, b_r \in \mathbb{Z}$

$$\left(X^{a_1} Y^{b_1} - 1\right)\left(X^{a_2} Y^{b_2} - 1\right) \cdots \left(X^{a_r} Y^{b_r} - 1\right) \in \text{Ann}(c)$$

CONSEQUENCES FOR NIVAT'S CONJECTURE

Theorem (Kari, Szabados, 2015)

If $\exists$ infinitely many $m, n \in \mathbb{N}, P_c(m, n) \leq mn$, then c is periodic.

Theorem (Szabados, 2018)

If $c = c_1 + c_2$ with c_1 and c_2 periodic, $\exists m, n \in \mathbb{N}, P_c(m, n) \leq mn$, then c is periodic.

LOW COMPLEXITY SUBSHIFTS

LOW COMPLEXITY SUBSHIFTS

A a set of $m \times n$ rectangular patterns

$$X_{\overline{A}} = \{c \mid \text{All patterns of } c \text{ are in } A\}$$

$X_{\overline{A}}$ **of low complexity**: $|A| \leq mn$ for some m, n

LOW COMPLEXITY SUBSHIFTS AND NIVAT

$X_{\overline{A}}$ of low complexity: $|A| \leq mn$ for some m, n

Then, for all $c \in X_{\overline{A}}$,

$$P_c(m, n) \leq mn$$

Conjecture (Nivat)

All $c \in X_F$ are periodic.

LOW COMPLEXITY SUBSHIFTS AND NIVAT

$X_{\overline{A}}$ of low complexity: $|A| \leq mn$ for some m, n

Then, for all $c \in X_{\overline{A}}$,

$$P_c(m,n) \leq mn$$

Theorem (Kari, M., 2020)

There exists $c \in X_F$ periodic (if $X_F \neq \emptyset$).

MAIN RESULT

Theorem (Kari, M., 2020)

If a subshift X contains a low complexity configuration, $\exists c \in X$ periodic.

in other words:

Theorem (Kari, M., 2020)

There is no aperiodic SFT containing a low-complexity configuration.

DOMINO PROBLEM FOR LOW COMPLEXITY SUBSHIFTS

In general, DP is **undecidable**.

Theorem (Kari, M., 2020)

DP is **decidable** for X of low complexity.

DP: DECIDING NON-EMPTINESS (LOW-COMPLEXITY)

Theorem (Kari, M., 2020)

There is a semi-algorithm deciding if $X_F \neq \emptyset$
for X_F of low complexity

1. `i=1`
2. `If a colorings of` B_i `with no` $p \in F$ `has a period:`

$$X_F \neq \emptyset$$

3. `Else i=i+1 and repeat`

WHAT NEXT ?

- → Nivat's conjecture
- → Domino problem still decidable for $P_c(m,n) \leq mn+1$?
  In other words: *Is there an aperiodic SFT of complexity $mn+1$?*
- → Higher dimensions ?

Thank you !

BONUS: $MN + 1$

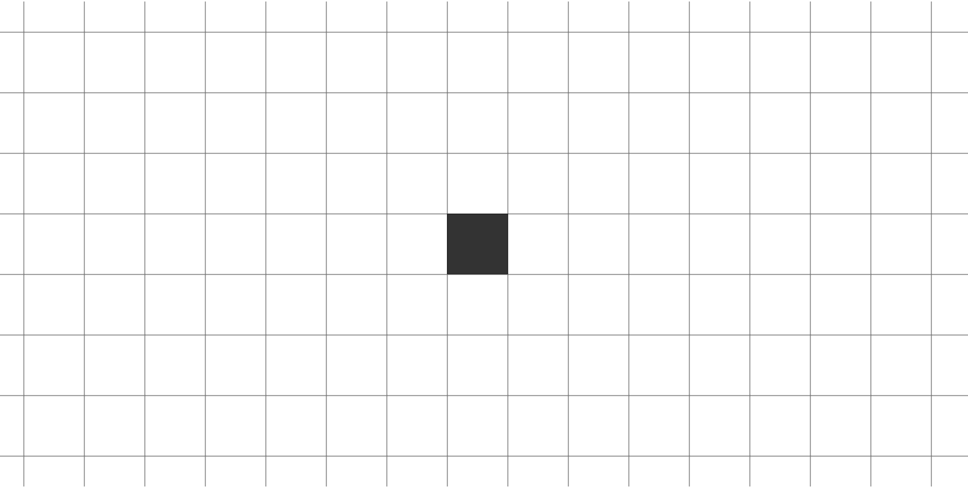
Nivat's conjecture does not holds for $mn + 1$ bound:

Theorem

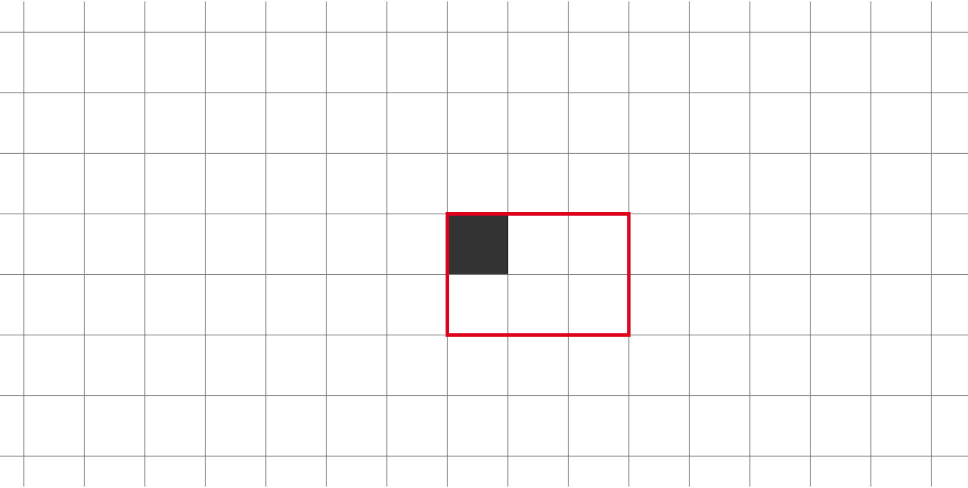
$$\exists c \in \mathcal{A}^{\mathbb{Z}^3},$$

$$\forall m, n > 0,\ P_c(m, n) = mn + 1 \quad \textit{and} \quad c \textit{ not periodic}$$

BONUS: $MN + 1$



BONUS: $MN + 1$



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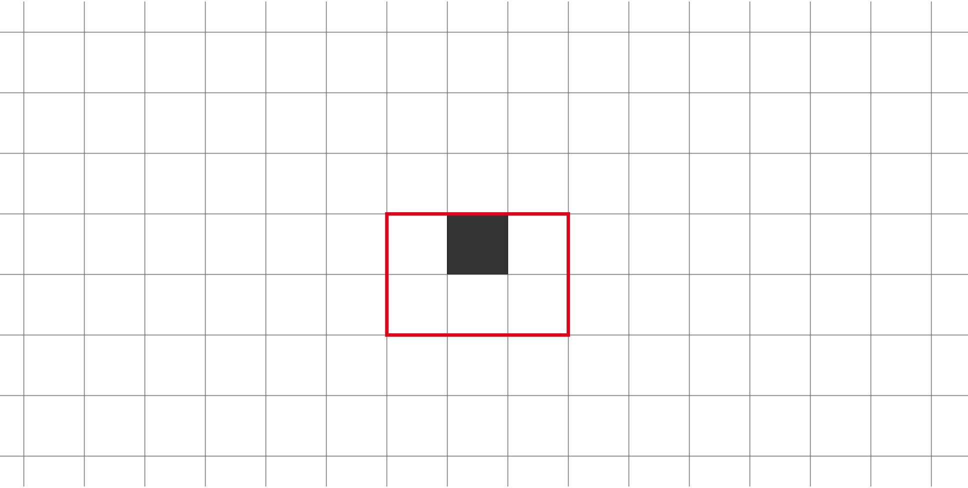
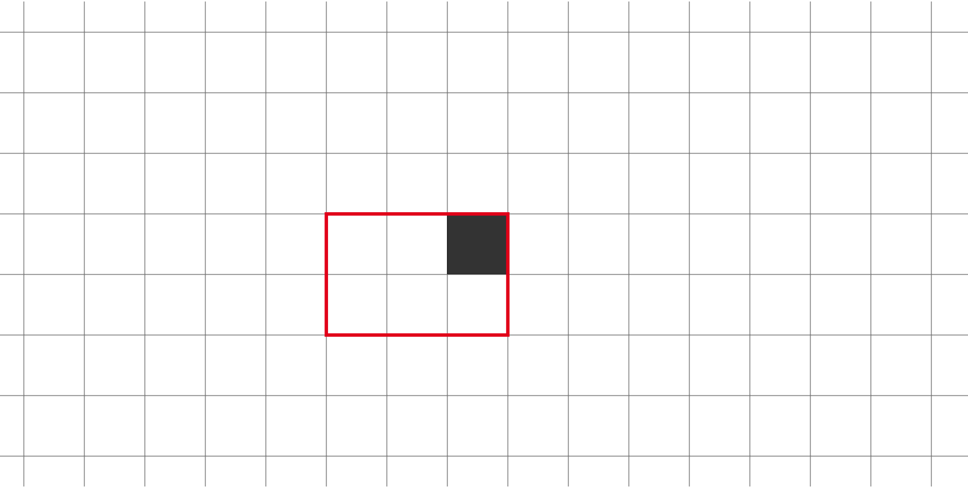
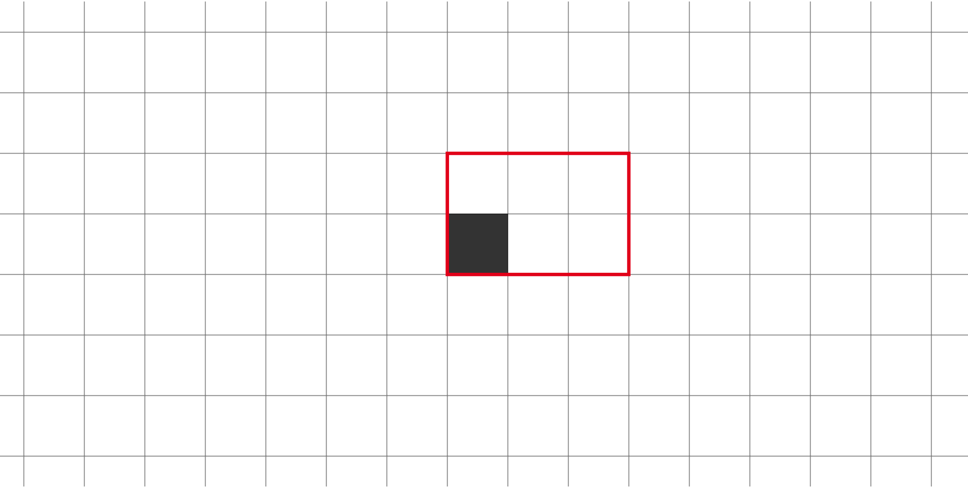


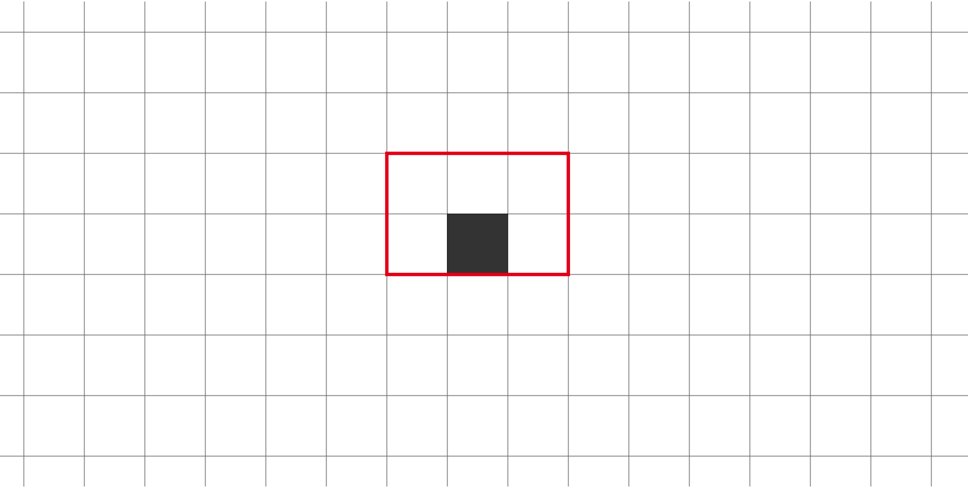
BONUS: $MN + 1$



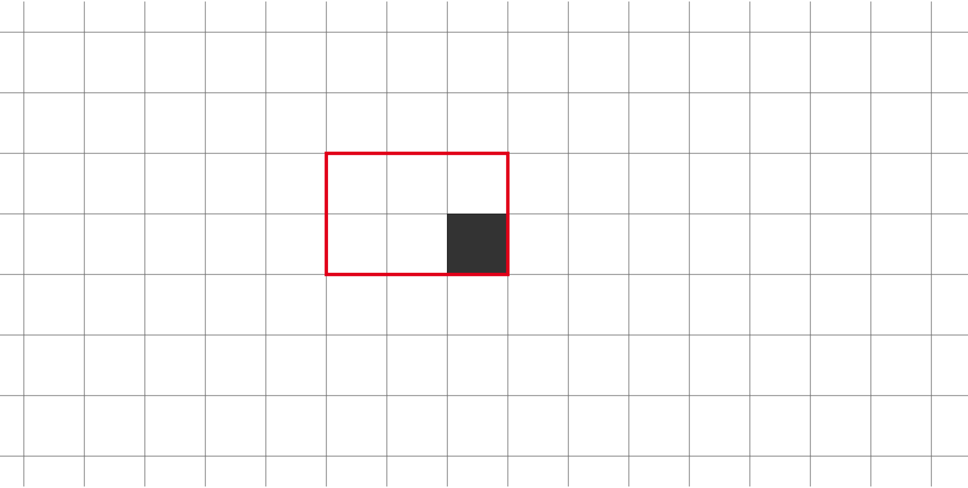
BONUS: $MN + 1$



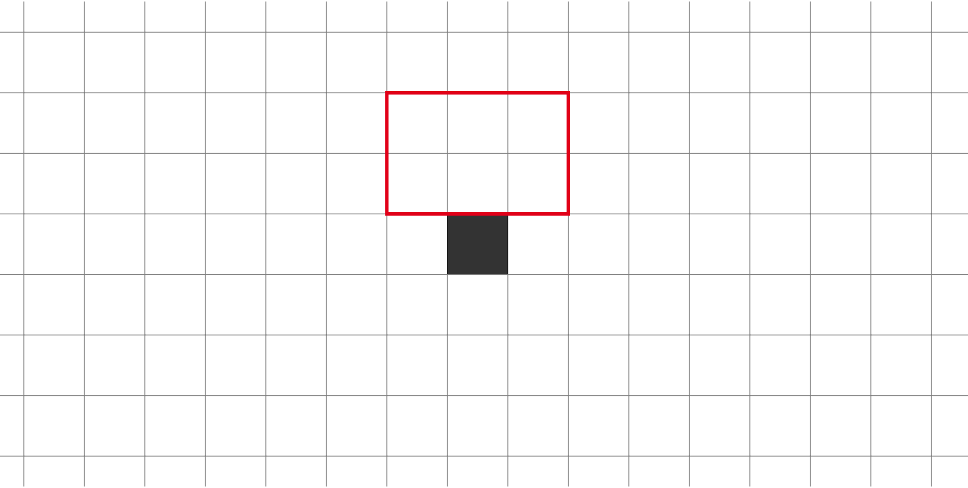
BONUS: $MN + 1$



BONUS: $MN + 1$



BONUS: $MN + 1$



BONUS: HIGHER DIMENSION

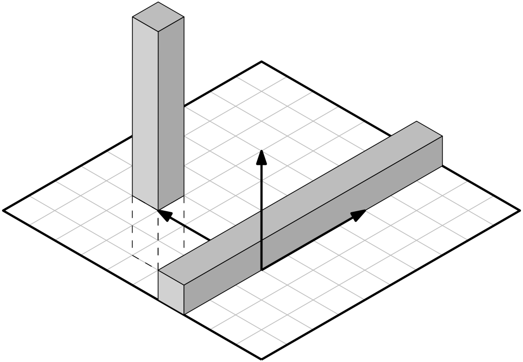
Nivat's conjecture does not holds in 3D and above:

Theorem

$$\exists c \in \mathcal{A}^{\mathbb{Z}^3},$$

$$\exists n > 0,\ P_c(n, n, n) \leq n^3 \quad \textit{and} \quad c \textit{ not periodic}$$

HIGHER DIMENSION ?



$$P_c(n, n, n) = 2n^2 + 1 < n^3$$

HSRM THEME

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